**Inequality**

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Let x,y,z be positive real numbers such that x + y + z ≥ 3 , prove that

$(x^3 + y^3 + z^3)(xyz + 2) \geq 9$.

**Solution by Arkady Alt , San Jose, California, USA.**

Let $F(x,y,z) := (x^3+y^3+z^3)(xyz+2)$. First note that suffices to prove inequality $F(x,y,z) \geq 9 = F(1,1,1)$ for $x,y,z > 0$ such that $x+y+z = 3$. Indeed, if $x+y+z > 3$ then for $x_1$ such that $\max\{0, 3-y-z\} < x_1 < x$ we have $F(x,y,z) > F(x_1,y,z)$.

Thus, the minimal value of $F(x,y,z)$ can't be attained in the point $(x,y,z)$ such that $x+y+z > 0$.

By replacing $(x,y,z)$ with $(3x,3y,3z)$ we obtain equivalent problem:

For $x,y,z > 0$ such that $x+y+z = 1$ prove inequality

(**1**) $3(x^3+y^3+z^3)(27xyz+2) \geq 1$ if.

Let $p := xy+yz+zx, q := xyz$. Then $x^3+y^3+z^3 = 1+3q-3p$ and inequality (**1**) becomes $3(1-3p+3q)(27q+2) \geq 1$.

Since $3p = xy+yz+zx \leq (x+y+z)^2 = 1$ then denoting $t := \sqrt{1-3p}$ we obtain $p = \dfrac{1-t^2}{3}$, where $t \in [0,1)$.

Noting that Vieta's System $\begin{cases} x+y+z = 1 \\ xy+yz+zx = \dfrac{1-t^2}{3} \\ xyz = q \end{cases}$ solvable* in non negative real $x,y,z$

iff $q_* \leq q \leq q^*$, where $q_* := \max\left\{0, \dfrac{(1+t)^2(1-2t)}{27}\right\}, q^* := \dfrac{(1-t)^2(1+2t)}{27}$

we obtain $3(1-3p+3q)(27q+2) - 1 = 3(3q+t^2)(27q+2) - 1 \geq 3(3q_*+t^2)(27q_*+2) - 1$.

If $t \in [0,1/2)$ then $3(3q_* - t^2)(27q_*+2) - 1 =$

$$\left(\frac{(1+t)^2(1-2t)}{3} + 3t^2\right)\left((1+t)^2(1-2t)+2\right) - 1 = \frac{t^2(15+4t^4-8t-18t^2-6t^3)}{3} \geq 0$$

because $15+4t^4-8t-18t^2-6t^3 \geq 15+4t^4-8\cdot\frac{1}{2}-18\cdot\left(\frac{1}{2}\right)^2-6\left(\frac{1}{2}\right)^3 = 4t^4+\frac{23}{4}$;

If $t > 1/2$ then $3(3q_*-t^2)(27q_*+2) - 1 = 3\left(1+3\cdot 0-3\cdot\dfrac{1-t^2}{3}\right)(27\cdot 0+2) - 1 =$ $6t^2 - 1 > 0$.

* Vieta's system

$$(\mathbf{V}) \quad \begin{cases} x+y+z = s \\ xy+yz+zx = p \\ xyz = q \end{cases}$$

solvable in real $x,y,z$ iff numbers $s,p,q$ satisfy inequality

(**B**) $p^2s^2 - 4p^3 + 18pqs - 4qs^3 - 27q^2 \geq 0$

(Sturm Theorem ( [1], p.6), and since $s = 1, p = \dfrac{1-t^2}{3}$ (**B**) can be represented in form

$$\frac{(1+t)^2(1-2t)}{27} \leq q \leq \frac{(1-t)^2(1+2t)}{27}.$$

1. D.S. Mitrinovic,J.E.Pecaric and V.Volenec, Recent advances in Geometric Inequalities.